COMPARATIVE STUDY OF THE ADIABATIC EVOLUTION OF A NONLINEAR DAMPED OSCILLATOR AND AN HAMILTONIAN GENERALIZED NONLINEAR OSCILLATOR

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Abstract

In this paper we study to what extent the canonical equivalence and the identity of the geometric phases of dissipative and conservative linear oscillators, established in a preceeding paper, can be generalized to nonlinear ones. Considering first the 1-D quartic generalized oscillator we determine, by means of a perturbative time dependent technic of reduction to normal forms, the canonical transformations which lead to the adiabatic invariant of the system and to the first order non linear correction to its Hannay angle. Then, applying the same transformations to the 1-D quartic damped oscillator we show that this oscillator is canonically equivalent to the linear generalized harmonic oscillator for finite values of the damping parameter (which implies no correction to the linear Hannay angle) whereas, in an appropriate weak damping limit, it becomes equivalent to the quartic generalized oscillator (which implies a non linear correction to this angle) .

INTRODUCTION The Hannay's angle [1] (classical counterpart of the Berry's geometric phase [2]), originally associated with the adiabatic evolution of classical hamiltonian systems, has been recently extended to a large class of dynamical equations corresponding to dissipative systems: non linear equations with limit cycles [3] or with more general internal symmetries [4], equations describing the dynamics of the laser [5], etc...In this context we have shown in a recent paper [6] that the simplest dissipative system, namely the damped harmonic oscillator specified by the time dependent Lagrangian

$$\mathcal{L}(q,\dot{q},\vec{\mu}) = \frac{1}{2}\, e^{2\int^t \lambda(s)\, ds}\, (\dot{q}^2 - \omega_o^2 q^2) \tag{1}$$

is canonically equivalent, even for time dependent parameters $(\lambda, \omega_o) \equiv \vec{\mu}$, to the generalized harmonic oscillator, a conservative system specified by the Hamiltonian

$$H(P,Q,\vec{\mu}) = \frac{P^2}{2} + \lambda PQ + \frac{\omega_o^2}{2} Q^2 \ . \tag{2}$$

(The generating function of the canonical transformation is $F(q,P,t) = qP\, e^{\int^t \lambda(s)\, ds}$). As a consequence the Hannay's angles of the two systems are identical. Their expression can be simply recovered, like in [7], using the transformed variable

$$Q = \frac{\partial F(q,P,t)}{\partial P} = q\, e^{\int^t \lambda(s)\, ds} \tag{3}$$

which brings the Lagrangian (1) to the form

$$L(Q,\dot{Q},\vec{\mu}) = \frac{1}{2}(\dot{Q}^2 - 2\lambda Q\dot{Q} - \omega^2 Q^2) \qquad (\omega^2 = \omega_o^2 - \lambda^2) \tag{4}$$

which also reads

$$L(Q,\dot{Q},\vec{\mu}) = \frac{1}{2}(\dot{Q}^2 - (\omega^2 - \dot{\lambda})Q^2) - \frac{1}{2}\frac{d}{dt}(\lambda Q^2) \ . \tag{5}$$

In this later expression the quantity $(\omega^2 - \dot{\lambda})$ is the square of the instantaneous frequency of the system. In the adiabatic limit where the parameters $\vec{\mu}$ are slowly time-varying functions $\vec{\mu}(\epsilon t)$ (with $\frac{\epsilon}{\omega_o} \ll 1$) one can make a first order

expansion of this instantaneous frequency with respect to the small adiabatic parameter ϵ. One gets the well known result [1,2]

$$\dot{\Theta} = \omega - \frac{\dot{\lambda}}{2\omega} \ . \tag{6}$$

where the 'dynamical' part ω of the time derivative $\dot{\Theta}$ of the phase of the oscillator appears to be corrected by an adiabatic contribution $-\dfrac{\dot{\lambda}}{2\omega}$ the integral of which in the parameters space is the 'geometrical' Hannay's angle.

The main purpose of this paper is to study to what extent the above results, concerning the canonical equivalence and the identity of the geometric phases of dissipative and conservative linear oscillators, can be generalized to nonlinear ones at least in the first order approximation of the perturbation theory. In the following we restrict ourselves to the quartic generalized oscillator and compare it with the quartic damped oscillator. We do not consider cubic terms because, at this order, they are non resonant *i.e.* without effect on the phase equation [8].

The first section is devoted to the quartic generalized oscillator the Hamiltonian of which is deduced from (2) by the addition of a term proportional to Q^4. In order to calculate the geometric phase of this system we extend the method of reduction to normal forms to the case where the parameters vary slowy with time. We are then able to solve perturbatively the Hamilton equations and to find the appropriate canonical transformations under the two assumptions of weak nonlinearity and of adiabaticity of the variation of the parameters. In particular we determine the adiabatic invariant I of the system and we find the expression of the first order nonlinear correction to the geometric Hannay part of the angle Θ. The reason for choosing the normal forms technic rather than the averaging method is that, besides the fact that it can in principle be developed to any order of perturbation (in the adiabatic regime), it is a standard approach for the study of nonlinear dissipative systems; it will allow us to deduce the appropriate canonical transformations in the dissipative case from the above ones.

The second section is devoted to the study of the damped quartic oscillator the Lagrangian of which is deduced from (1) by the addition of a term proportional to q^4. Using the time-dependent canonical transformations found in section 1 and keeping the same hypotheses of weak nonlinearity and of adiabaticity we obtain the following results. For finite values of the damping

parameter λ, *i.e.* when resonance does not exist, the quartic term is without effect on the geometric part of the angle. In that case the quartic damped oscillator can be shown to be canonically equivalent to the linear generalized harmonic oscillator. However, there also exists a weak damping limit characterized by a magnitude of λ going to zero with the adiabatic parameter in such a way that the resonance phenomenon resurges in the limit. In this limit the quartic term contributes and one recovers the Hannay's angle of the quartic generalized oscillator determined in the previous section. This later result generalizes to nonlinear systems the result established in [6] for linear oscillators.

1. GENERALIZED QUARTIC OSCILLATOR The simplest nonlinear extension of the generalized harmonic oscillator leading to a resonant term in the equation of motion is obtained by adding a quartic term to the Hamiltonian (2) which thus becomes:

$$H_G(P,Q,\vec{\mu}) = \frac{P^2}{2} + \lambda PQ + \frac{\omega_o^2}{2}Q^2 + \frac{\nu}{4}Q^4. \tag{7}$$

The Hamilton equations for Q and P read

$$\dot{Q} = \lambda Q + P \quad , \quad \dot{P} = -\lambda P - \omega_0^2 Q - \nu Q^3 \ . \tag{8}$$

In order to solve these nonlinear coupled equations it is convenient to introduce, in place of P and Q, the complex variable z, and its complex conjugate $\overline{z}$, defined by

$$z = \sqrt{\frac{\omega}{2}}\Big[Q - \frac{i}{\omega}\left(\lambda Q + P\right)\Big], \qquad (\omega^2 = \omega_o^2 - \lambda^2) \ . \tag{9}$$

Equivalently Q and P read:

$$Q = \frac{1}{\sqrt{2\omega}}(z+\overline{z}), \qquad P = \frac{i\omega - \lambda}{\sqrt{2\omega}}z - \frac{i\omega + \lambda}{\sqrt{2\omega}}\overline{z} \ . \tag{10}$$

For time dependent parameters $(\lambda, \omega_0, \nu) \equiv \vec{\mu}$, the Hamilton equations for Q and P lead to the following nonlinear equation for z:

$$\dot{z} = i\omega z + \frac{i\nu}{4\omega^2}(z+\overline{z})^3 - \frac{i\dot{\lambda}}{2\omega}z + \frac{\dot{\omega} - i\dot{\lambda}}{2\omega}\overline{z}. \tag{11}$$

(Note that the three parameters λ, ω and ν do not play the same role: the time derivative $\dot{\nu}$ of the parameter associated with the non linear quartic term does not appear in (11) in contradistinction with $\dot{\lambda}$ and $\dot{\omega}$.) This equation can be solved perturbatively using its canonical reduction to normal form [9] if one assumes that the system is weakly nonlinear ($\frac{\nu}{\omega_o^2} Q^2 \ll 1$) and that the parameters vary adiabatically ($\vec{\mu}$ is a slowly time varying function $\vec{\mu}(\epsilon t)$ with $\frac{\epsilon}{\omega_o} \ll 1$). Under these conditions, let us introduce the near to identity transformation

$$z = u + \delta \overline{u} \tag{12}$$

in order to eliminate the nonresonant term proportional to $\overline{u}$ into the equation for the new variable u. For $\delta = \frac{\dot{\lambda} + i\dot{\omega}}{4\omega^2}$ (δ small, of order ϵ) the coefficient of $\overline{u}$ cancels and the equation for u reads:

$$\dot{u} = i\Big(\omega - \frac{\dot{\lambda}}{2\omega}\Big)u + \frac{i\nu}{4\omega^2}(u+\overline{u})^2\Big[(1+\delta+3\overline{\delta})u + (1+4\delta)\overline{u}\Big] \ . \tag{13}$$

The first term in (13) already exhibits the Hannay's angle of the linear generalized oscillator. In order to obtain the non linear correction one must introduce a second (near to identity) change of variable

$$u = v + \alpha v^3 + \beta v \overline{v}^2 + \gamma \overline{v}^3. \tag{14}$$

The requirement that the equation for v no longer contains the nonresonant cubic terms present in (13) leads to differential equations for the time dependent coefficients α, β and γ of the transformation. These differential equations are explicitly written and their solutions discussed in the appendix. The resulting equation for v, valid up to the first order in ϵ and in the weak nonlinearity parameter's ν, is:

$$\dot{v} = i(\omega - \frac{\dot{\lambda}}{2\omega})v + \frac{3i\nu}{4\omega^2}(1 + \frac{\dot{\lambda}}{\omega^2})v^2\overline{v} \tag{15}$$

Then, setting $v = A\, e^{i\Theta}$, the equations for the amplitude A and for the angle Θ read:

$$\dot{A} = 0, \tag{16}$$

$$\dot{\Theta} = \omega\Big(1 + \frac{3\nu}{4\omega^3}A^2\Big) - \frac{\dot{\lambda}}{2\omega}\Big(1 - \frac{3\nu}{2\omega^3}A^2\Big). \tag{17}$$

Equation (16) shows that A is an adiabatic invariant related, as seen below, to the action I of the system

$$I = A^2 \ . \tag{18}$$

The first term in the r.h.s. of equation (17) accounts for the well known result that the quartic term $\frac{\nu}{4}Q^4$ in (7), responsible of the presence of the resonant $v^2\overline{v}$ terms in (15), induces a renormalization of the linear frequency ω which becomes

$$\Omega = \omega(1 + \frac{3\nu}{4\omega^3}\, A^2) \ . \tag{19}$$

(The facts that the frequency does not explicitly appear in the expression of I and that the correction term $\frac{3\nu}{4\omega^2}\, A^2$ to the linear frequency is different from the expression found in reference [8] are due to the use of the amplitude A of the transformed variable v in (18) and (19) in place of the amplitude a of the original variable Q; in particular, the usual expression for Ω in terms of a, ω and ν can be recovered from (19) noting that $A = \sqrt{\frac{\omega}{2}}\, a$.) The second term $\frac{\dot{\lambda}}{2\omega}\Big(-1 + \frac{3\nu}{2\omega^2}A^2\Big)$ in the r.h.s. of equation (17), which exists only for time dependent parameters, is the geometric, non integrable, Hannay's part of $\dot{\Theta}$. Like the dynamical part Ω it also contains a contribution from the nonlinear resonant term. However, as this geometrical part is defined up to a total time derivative one can write it, for ν constant, under the same form as in the linear case $\Big(-\frac{\dot{\lambda}}{2\omega}\Big)$ in terms of the renormalized frequency Ω and of a renormalized damping parameter Λ defined by

$$\Lambda = \lambda\Big(1 - \frac{3\nu}{\omega^3}\, A^2\Big) \ . \tag{20}$$

Indeed, the equality $\frac{\dot{\Lambda}}{2\Omega} = \frac{\dot{\lambda}}{2\omega}(1 - \frac{3\nu}{2\omega^3}\, A^2) - \frac{d}{dt}\Big(\frac{9\nu\lambda}{8\omega^4}\, A^2\Big)$ is exact up to the first order in the weak nonlinear term proportional to νA^2. Consequently, the equation (17) for the angle variable Θ takes a functional form,

$$\dot{\Theta} = \Omega - \frac{\dot{\Lambda}}{2\Omega} \ , \tag{21}$$

identical to the one, $\dot{\Theta} = \omega - \frac{\dot{\lambda}}{2\omega}$, obtained in the linear case. One can say that, for ν constant, the effect of the weak quartic nonlinearity on the phase (the 'geometrical' as well as the'dynamical' parts) amounts to a simple renormalization of both ω and λ.

We now show that the dynamical variables (I, Θ) and (P, Q) are related by a time dependent canonical transformation and that the equations (16-18) are the Hamilton equations for action-angle variables associated with the time dependent Hamiltonian $\mathcal{H}_G(I, \Theta, t)$:

$$\mathcal{H}_G(I, \Theta, t) = I\omega\Big(1 - \frac{\dot{\lambda}}{2\omega^2}\Big) + I^2\frac{3\nu}{8\omega^2}\Big(1 + \frac{\dot{\lambda}}{\omega^2}\Big) \ . \tag{22}$$

This can be verified by inspection of the set of successive transformations $(P, Q) \rightarrow (-i\overline{z}, z) \rightarrow (iu, \overline{u}) \rightarrow (-i\overline{v}, v) \rightarrow (I, \Theta)$, which will be also of importance in section 2. As concerns the first transformation $(P, Q) \rightarrow (-i\overline{z}, z)$ its generating function $F(Q, z)$ is obtained by integration of the equations $\frac{\partial F}{\partial Q} = P(Q, z)$ and $\frac{\partial F}{\partial z} = i\overline{z}(Q, z)$ deduced from the differential identity (characterizing a canonical transformation)

$$PdQ - H_G dt = -i\overline{z}dz - \mathcal{H}^z_G dt + dF \ . \tag{23}$$

Taking into account the relations (9-10) one gets

$$F(Q, z) = -\frac{1}{2}(i\omega + \lambda)Q^2 + i\sqrt{2\omega}Qz - \frac{1}{2}iz^2 \ . \tag{24}$$

The Hamiltonian $\mathcal{H}^z_G$ for the new conjugate variables $(-i\overline{z}, z)$ is then obtained from the relation $\mathcal{H}^z_G = H_G + \frac{\partial F}{\partial t}$. Its expression reads

$$\mathcal{H}^z_G = \omega z\overline{z} + \frac{\nu}{16\omega^2}(z + \overline{z})^4 + \frac{i\dot{\omega}}{4\omega}(z^2 - \overline{z}^2) - \frac{\dot{\lambda}}{4\omega}(z + \overline{z})^2 \tag{25}$$

and one can verify that the Hamilton equations

$$\dot{z} = \frac{\partial \mathcal{H}^z_G}{\partial(-i\overline{z})} \quad , \quad -i\dot{\overline{z}} = -\frac{\partial \mathcal{H}^z_G}{\partial z} \tag{26}$$

indeed coincide with the equation (11) for z and the corresponding one for $\overline{z}$.

The transformation $(-i\overline{z}, z) \rightarrow (iu, \overline{u})$ associated with the generating function

$$G(z, \overline{u}) = -iz\overline{u} + \frac{1}{2}i\delta\overline{u}^2 - \frac{1}{2}\overline{\delta}z^2 \tag{27}$$

is also canonical. The expression of the Hamiltonian $\mathcal{H}_G^u$ for the new conjugate variables $(iu, \overline{u})$ is obtained from the relation $\mathcal{H}_G^u = \mathcal{H}_G^z + \dfrac{\partial G}{\partial t}$ and, in this hamiltonian formulation of the reduction to normal form, δ and $\overline{\delta}$ are determined by the requirement that $\mathcal{H}_G^u$ no longer contains terms proportional to $\overline{u}^2$ and u^2. This is equivalent to the above requirement that the equation for u (resp. $\overline{u}$) no longer contains non resonant term proportional to $\overline{u}$ (resp u) and it leads to the same value $\dfrac{\dot{\lambda} + i\dot{\omega}}{4\omega^2}$ for δ. Then, $\mathcal{H}_G^u$ reads

$$\mathcal{H}_G^u = (\omega - \frac{\dot{\lambda}}{2\omega})u\overline{u} + \frac{\nu}{16\omega^2}(u + \overline{u})^4 + \frac{\nu}{4\omega^2}(u + \overline{u})^3(\delta\overline{u} + \overline{\delta}u) \ . \tag{28}$$

(In fact $\dfrac{\partial G}{\partial t}$ does not contribute to this expression of $\mathcal{H}_G^u$ valid up to the first order in ϵ since it only contains terms proportional to $\dot{\delta}$ and $\dot{\overline{\delta}}$ and is thus of order ϵ^2.)

In the same way, the transformation $(iu, \overline{u}) \rightarrow (-i\overline{v}, v)$ associated with the generating function

$$K(\overline{u}, v) = iv\overline{u} + \alpha v^3\overline{u} + \frac{1}{3}\beta v\overline{u}^3 + \frac{1}{4}\gamma\overline{u}^4 - \frac{1}{4}\overline{\gamma}v^4 \tag{29}$$

is canonical. The values of α, β and γ, now determined by imposing that the Hamiltonian $\mathcal{H}_G^v = \mathcal{H}_G^u + \dfrac{\partial K}{\partial t}$ for the new conjugate variables $(-i\overline{v}, v)$ does not contain quartic non-resonant terms, are those found in the appendix; the expression of $\mathcal{H}_G^v$ is

$$\mathcal{H}_G^v = (\omega - \frac{\dot{\lambda}}{2\omega})v\overline{v} + \frac{3\nu}{8\omega^2}(1 + \frac{\dot{\lambda}}{\omega^2})v^2\overline{v}^2 \ . \tag{30}$$

Finally the transformation $(-i\overline{v}, v) \rightarrow (I, \Theta)$ associated with the generating function

$$M(v, \Theta) = -\frac{1}{2}v^2\exp(-2i\Theta) \tag{31}$$

is canonical and the Hamiltonian (30) transforms into (22) as announced. This expression (22) shows that $I = A^2$ is the action of the system and allows to determine the adiabatic invariant as a function of the energy. 2.

DAMPED QUARTIC OSCILLATOR

The damped quartic oscillator which we consider in this section is described by the equation:

$$\ddot{q} + \omega_0^2 q + 2\lambda\dot{q} + \nu q^3 = 0. \tag{32}$$

It can be obtained from the generalized time-dependent Caldirola-Kanai Lagrangian [10]

$$\mathcal{L}_D(q, \dot{q}, \vec{\mu}) = \frac{1}{2} e^{2\int^t \lambda(s)\, ds} (\dot{q}^2 - \omega_0^2 q^2 - \frac{1}{2}\nu q^4). \tag{33}$$

In terms of the new variable

$$Q = q e^{\int^t \lambda(s)\, ds} \tag{34}$$

the Lagrangian (33) reads

$$L_D(Q, \dot{Q}, \vec{\mu}) = \frac{1}{2}(\dot{Q}^2 - 2\lambda Q\dot{Q} - \omega^2 Q^2 - \frac{1}{2}\nu Q^4 e^{-2\int^t \lambda(s)\, ds}) \tag{35}$$

and the corresponding time-dependent Hamiltonian $H_D(P, Q, \vec{\mu})$, where $P = \dfrac{\partial L_D}{\partial \dot{Q}} = \dot{Q} - \lambda Q$, takes a form :

$$H_D(P, Q, \vec{\mu}) = \frac{P^2}{2} + \lambda P Q + \frac{\omega_0^2}{2} Q^2 + \frac{\nu}{4} Q^4 e^{-2\int^t \lambda(s)\, ds} \tag{36}$$

identical to the expression (7) for the Hamiltonian of the quartic generalized oscillator, except the crucial exponential factor in the quartic term. Then, making the same successive changes of variables as in the previous section, namely $(P, Q) \to (-i\overline{z}, z) \to (iu, \overline{u}) \to (-i\overline{v}, v)$, we obtain without any new calculation the following Hamiltonian for the conjugate variables $(-i\overline{v}, v)$:

$$\mathcal{H}_D^v = (\omega - \frac{\dot{\lambda}}{2\omega}) v\overline{v} + \frac{3\nu}{8\omega^2}(1 + \frac{\dot{\lambda}}{\omega^2}) v^2 \overline{v}^2 \ e^{-2\int^t \lambda(s)\, ds} \tag{37}$$

At this point a remark is of order. In contradistinction with the generalized quartic oscillator the damped quartic oscillator does not exhibit the resonance phenomenon. The theory of normal forms teaches that it is possible in that case to eliminate non linear terms. As a consequence one expects (as announced in [6]) that the Hannay's angle does no get any non linear contribution. Let us see how this comes in the formalism of canonical transformations. To this end we introduce the (near to identity) change of variable

$$v = w + i\sigma w^2 \overline{w}. \tag{38}$$

The transformation $(-i\overline{v}, v) \to (iw, \overline{w})$ is canonical for σ real. It corresponds to the generating function

$$N(v, \overline{w}) = -iv\overline{w} - \frac{1}{2}\sigma v^2 \overline{w}^2 \tag{39}$$

and the transformed Hamiltonian $\mathcal{H}_D^w = \mathcal{H}_D^v + \dfrac{\partial N}{\partial t}$ reads

$$\mathcal{H}_D^w = (\omega - \frac{\dot{\lambda}}{2\omega}) w\overline{w} - \frac{1}{2}\Big(\dot{\sigma} - \frac{3\nu}{4\omega^2}(1 + \frac{\dot{\lambda}}{\omega^2})\, e^{-2\int^t \lambda(s)\, ds}\Big) w^2 \overline{w}^2 \ . \tag{40}$$

The second term in the r.h.s. of (40) can be set equal to zero. Indeed, because of the presence of the exponential factor the corresponding value of σ remains small (proportional to ν) which guarantees that the transformation (38) is close to identity. Thus for λ finite the quartic damped oscillator is canonically equivalent to the generalized harmonic oscillator and the phases of the two systems are identical. Note that such an elimination can not be done on the Hamiltonian (30) because the integration of $\dot{\sigma}$ would lead to unbounded terms proportional to the time t.

Let us now consider the case where the magnitude of the damping parameter λ is close to zero in such a way that the resonance phenomenon cannot be avoided. More precisely let λ be of the form

$$\lambda(t) = \epsilon^a \tilde{\lambda}(\epsilon^b t) \tag{41}$$

where a and b are positive numbers such that $a > b > 0$ and $a + b = 1$. b positive ensures the validity of the adiabatic hypothesis for λ and $a + b = 1$ that $\dot{\lambda}$ remains of order ϵ, like $\dot{\omega}$. For such a behaviour of λ the integral

$\int^t \lambda(s)\, ds$ is of order ϵ^{a-b} with $(a-b)>0$. Then, replacing the exponential factor in (37) by the relevant terms of its expansion one gets the following expression (valid up to the first order in ϵ) for the Hamiltonian $\mathcal{H}_D^v$:

$$\mathcal{H}_D^v = (\omega - \frac{\dot\lambda}{2\omega})v\overline{v} + \frac{3\nu}{8\omega^2}(1 - 2\int^t \lambda(s)\, ds + \frac{\dot\lambda}{\omega^2})v^2\overline{v}^2 \ . \tag{42}$$

For the same reasons as for the Hamiltonian (30) the non linear terms in (42) cannot be eliminated. It is interesting to note that the term proportional to the time derivative of the parameters, $-\frac{\dot\lambda}{2\omega}v\overline{v} + \frac{3\nu\dot\lambda}{8\omega^4}v^2\overline{v}^2$, is identical in (30) and (42). Therefore in this weak damping limit the Hannay's angle of the quartic damped oscillator is the same as the one of the quartic generalized harmonic oscillator. Due to the presence of the supplementary dynamical term $-\frac{3\nu}{4\omega^2}\int^t \lambda(s)\, ds\, v^2\overline{v}^2$ in (42) the two systems seem to be not canonically equivalent. However one can express the Hamiltonian (42) in terms of the renormalized parameter $\tilde\nu = \nu(1-2\int^t\lambda(s)\, ds)$ (a change which, in the degree of accuracy of our calculations, does not affect the geometrical nonlinear part of the angle) to get an expression

$$\mathcal{H}_D^v = (\omega - \frac{\dot\lambda}{2\omega})v\overline{v} + \frac{3\tilde\nu}{8\omega^2}(1 + \frac{\dot\lambda}{\omega^2})v^2\overline{v}^2 \tag{43}$$

similar to (30). The damped quartic oscillator with parameters ω_0, λ and ν is thus, in this weak damping limit and in the first order approximation of perturbation theory, canonically equivalent to the generalized quartic oscillator with parameters ω_0, λ and $\tilde\nu$ and both systems have identical Hannay's angles. These results are the generalization for nonlinear oscillators of the result established in [6] for linear ones.

APPENDIX

The nonlinear, near to identity, change of variable (14) transforms the equation (13) for u into the following equation for the new variable v:

$$\dot v = i\Big(\omega - \frac{\dot\lambda}{2\omega}\Big)v + \frac{3i\nu}{4\omega^2}(1+2(\delta+\overline{\delta}))\, v^2\overline{v} - D_1\alpha\ v^3 - D_2\beta\ v\overline{v}^2 - D_3\gamma\ \overline{v}^3 \quad (A.1)$$

where the differential operators D_1, D_2 and D_3 are such that

$$D_1\alpha = \dot\alpha + 2i\Big(\omega - \frac{\dot\lambda}{2\omega}\Big)\alpha - \frac{i\nu}{4\omega^2}(1+\delta+3\overline{\delta}) \quad (A.2)$$

$$D_2\beta = \dot\beta - 2i\Big(\omega - \frac{\dot\lambda}{2\omega}\Big)\beta - \frac{3i\nu}{4\omega^2}(1 + 3\delta + \overline{\delta}) \qquad (A.3)$$

$$D_3\gamma = \dot\gamma - 4i\Big(\omega - \frac{\dot\lambda}{2\omega}\Big)\gamma - \frac{i\nu}{4\omega^2}(1 + 4\delta) \qquad (A.4)$$

For α solution of $D_1\alpha = 0$, β solution of $D_2\beta = 0$ and γ solution of $D_3\gamma = 0$ the nonresonant cubic terms can be eliminated in (A.1). The three differential equations having the same structure, it is sufficient to study the behaviour of the solution of one of them, for example α. Since $\delta = \dfrac{\dot\lambda + i\dot\omega}{4\omega^2}$, the differential equation for α reads

$$\dot\alpha + 2i\Big(\omega - \frac{\dot\lambda}{2\omega}\Big)\alpha - \frac{i\nu}{4\omega^2}(1 + \frac{2\dot\lambda - i\dot\omega}{2\omega^2}) = 0 \ . \qquad (A.5)$$

This equation which contains terms of order zero and of order one with respect to the small adiabatic parameter ϵ can be solved by a perturbative method and its solution can be written under the form of an expansion with respect to ϵ. Neglecting the terms of order ϵ in (A.5) (*i.e.* all the time derivatives) one finds the zero order approximate solution $\alpha_0 = \dfrac{\nu}{8\omega^3}$. Then putting $\alpha = \alpha_0 + \alpha_1$ into (A.5) one gets an equation for α_1 which contains terms of order one and of order two with respect to ϵ. Neglecting the terms proportional to ϵ^2 in this equation, one obtains an expression (of order ϵ) for α_1 and so on. One verifies that α, and thus also β and γ, are small for weak nonlinearity and in the adiabatic hypothesis (the leading terms are proportional to ν and ϵ). This validades the above calculations which suppose that the transformation is near to the identity. Finally the nonresonant cubic terms can indeed be eliminated in (A.1) which reduces to (15).

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